

Amalgamated duplication of the Banach algebra \mathfrak{A} along a \mathfrak{A} -bimodule \mathcal{A}

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Abstract

Let \mathcal{A} and \mathfrak{A} be Banach algebras such that \mathcal{A} is a Banach \mathfrak{A} -bimodule with compatible actions. We define the product $\mathcal{A} \rtimes \mathfrak{A}$, which is a strongly splitting Banach algebra extension of \mathfrak{A} by \mathcal{A} . After characterization of the multiplier algebra, topological centre, (maximal) ideals and spectrum of $\mathcal{A} \rtimes \mathfrak{A}$, we restrict our investigation to the study of semisimplicity, regularity, Arens regularity of $\mathcal{A} \rtimes \mathfrak{A}$ in relation to that of the algebras \mathcal{A} , \mathfrak{A} and the action of \mathfrak{A} on \mathcal{A} . We also compute the first cohomology group $H^1(\mathcal{A} \rtimes \mathfrak{A}, (\mathcal{A} \rtimes \mathfrak{A})^{(n)})$ for all $n \in \mathbb{N} \cup \{0\}$ as well as the first-order cyclic cohomology group $H_\lambda^1(\mathcal{A} \rtimes \mathfrak{A}, (\mathcal{A} \rtimes \mathfrak{A})^{(1)})$, where $(\mathcal{A} \rtimes \mathfrak{A})^{(n)}$ is the n -th dual space of $\mathcal{A} \rtimes \mathfrak{A}$ when $n \in \mathbb{N}$ and $\mathcal{A} \rtimes \mathfrak{A}$ itself when $n = 0$. These results are not only of interest in their own right, but also they pave the way for obtaining some new results for Lau products and module extensions of Banach algebras as well as triangular Banach algebra. Finally, special attention is devoted to the cyclic and n -weak amenability of $\mathcal{A} \rtimes \mathfrak{A}$.

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1 Introduction

Let \mathcal{A} and \mathcal{B} be Banach algebras. It is well-known that the Cartesian product space $\mathcal{A} \times \mathcal{B}$ equipped with the ℓ^1 -norm and coordinatewise operations is a Banach algebra. In order to provide new examples of Banach algebras as well as a wealth of (counter) examples in different branches of functional analysis, the construction of an algebra product on the Cartesian product space $\mathcal{A} \times \mathcal{B}$ to make it a Banach algebra has been studied in a series of papers recently. The first important paper related to this construction is Lau's paper [16] which he defined a new algebra product on the Cartesian product space $\mathcal{A} \times \mathcal{B}$ for the case where \mathcal{B} is the predual of a van Neumann algebra \mathcal{M} such that the identity of \mathcal{M} is a multiplicative linear functional on \mathcal{B} . Later on, Monfared [18] extended the Lau product to arbitrary Banach algebras. This construction has many applications in different contexts, see for examples [9, 15, 19, 20]. Moreover, it is notable that such constructions are also investigated in commutative ring theory and have been extensively studied in recent years, see for examples [6, 7, 8, 12] and the references therein.

On the other hand, motivated by Wedderburn's principal theorem [1], splitting of Banach algebra extensions has been a major question in the theory of Banach algebras and several researchers have used the splitting of Banach algebra extensions as a tool for the study of Banach

algebras. For example, module extensions as generalizations of Banach algebra extensions were introduced and first studied by Gourdeau [13] were used to show that amenability of $\mathcal{A}^{(2)}$, the second dual space of \mathcal{A} , implies amenability of \mathcal{A} ; Filali and Eshaghi Gordji [10] used triangular Banach algebras to answer some questions asked by Lau and Ülger [17]; Zhang [22] used module extensions to construct an example of a weakly amenable Banach algebra which is not 3-weakly amenable. For some other applications of splitting of Banach algebra extensions we refer the reader to the references [2, 3, 4, 11, 21].

This paper continues these investigations. In details, the product $\mathcal{A} \rtimes \mathfrak{A}$ of Banach algebras defined in this paper (Definition 3.1 below) and we point out that the Lau products and module extension of Banach algebras as well as triangular Banach algebras can be regarded as examples of $\mathcal{A} \rtimes \mathfrak{A}$ (Example 3.3 below) and many basic properties can be derived within this more general context. Apart from the determination of the action of $\mathcal{A} \rtimes \mathfrak{A}$ on its n -th dual space $(\mathcal{A} \rtimes \mathfrak{A})^{(n)}$ ($n \in \mathbb{N}$), we give a characterization of the spectrum of $\mathcal{A} \rtimes \mathfrak{A}$. Sufficient and necessary conditions for semisimplicity, regularity, Arens regularity and strong Arens irregularity of $\mathcal{A} \rtimes \mathfrak{A}$ are given. Moreover, we obtain characterizations of multiplier algebra, topological centre, (maximal) ideals of $\mathcal{A} \rtimes \mathfrak{A}$ and compute the first cohomology group $H^1(\mathcal{A} \rtimes \mathfrak{A}, (\mathcal{A} \rtimes \mathfrak{A})^{(n)})$ ($n \in \mathbb{N} \cup \{0\}$) as well as the first-order cyclic cohomology group $H_\lambda^1(\mathcal{A} \rtimes \mathfrak{A}, (\mathcal{A} \rtimes \mathfrak{A})^{(1)})$.

2 Preliminaries

Throughout this paper, for any Banach space \mathcal{Y} and integer number $n \geq 0$, by $\mathcal{Y}^{(n)}$ we denote the n -th dual space of \mathcal{Y} when $n \geq 1$ and \mathcal{Y} itself when $n = 0$. Moreover, we always consider \mathcal{Y} as naturally embedded into $\mathcal{Y}^{(2)}$.

Let \mathcal{C} be a Banach algebra and let $\sigma(\mathcal{C})$ stands for the spectrum of \mathcal{C} ; that is, the set of all non-zero multiplicative linear functionals on \mathcal{C} . If \mathcal{C} is commutative and $c \in \mathcal{C}$, then we define $\hat{c} : \sigma(\mathcal{C}) \rightarrow \mathbb{C}$ by $\hat{c}(\varphi) = \varphi(c)$ for all $\varphi \in \sigma(\mathcal{C})$. The continuous function \hat{c} is called the Gelfand transform of c . It is easily checked that the mapping

$$\Gamma_{\mathcal{C}} : \mathcal{C} \rightarrow C_0(\sigma(\mathcal{C})); \quad c \mapsto \hat{c}$$

is a homomorphism which is called the Gelfand homomorphism of \mathcal{C} . In this case, we say that \mathcal{C} is semisimple if the Gelfand homomorphism is injective and it is called regular if for each closed subset E of $\sigma(\mathcal{C})$ and $\varphi_0 \in \sigma(\mathcal{C}) \setminus E$, there exists $c \in \mathcal{C}$ such that $\varphi_0(c) \neq 0$ and $\varphi(c) = 0$ for all $\varphi \in E$. Recall also that on $\mathcal{C}^{(2)}$ there exist two natural products extending the one on \mathcal{C} , known as the first and second Arens products of $\mathcal{C}^{(2)}$ where the first Arens product “ \circ ” on $\mathcal{C}^{(2)}$ defined in three steps as follows:

$$\begin{aligned} \langle a^{(2)} \circ b^{(2)}, a^{(1)} \rangle &= \langle a^{(2)}, b^{(2)} \circ a^{(1)} \rangle, \\ \langle b^{(2)} \circ a^{(1)}, a \rangle &= \langle b^{(2)}, a^{(1)} \circ a \rangle, \\ \langle a^{(1)} \circ a, b \rangle &= \langle a^{(1)}, ab \rangle, \end{aligned}$$

where $a, b \in \mathcal{C}$, $a^{(1)} \in \mathcal{C}^{(1)}$ and $a^{(2)}, b^{(2)} \in \mathcal{C}^{(2)}$; similarly, by using symmetry, the second Arens product “ Δ ” on $\mathcal{C}^{(2)}$ is defined as follows:

$$\langle a^{(2)} \Delta b^{(2)}, a^{(1)} \rangle = \langle b^{(2)}, a^{(1)} \Delta a^{(2)} \rangle,$$

$$\begin{aligned}\langle a^{(1)} \triangle a^{(2)}, a \rangle &= \langle a^{(2)}, a \triangle a^{(1)} \rangle, \\ \langle a \triangle a^{(1)}, b \rangle &= \langle a^{(1)}, ba \rangle,\end{aligned}$$

The first topological centre

$$\mathfrak{Z}_t(\mathcal{C}^{(2)}) := \left\{ a^{(2)} \in \mathcal{C}^{(2)} : a^{(2)} \circ b^{(2)} = a^{(2)} \triangle b^{(2)} \text{ for all } b^{(2)} \in \mathcal{C}^{(2)} \right\}$$

of $\mathcal{C}^{(2)}$ is a closed subalgebra of $(\mathcal{C}^{(2)}, \circ)$ containing \mathcal{C} . The algebra \mathcal{C} is called Arens regular [respectively, strongly Arens irregular] if $\mathfrak{Z}_t(\mathcal{C}^{(2)}) = \mathcal{C}^{(2)}$ [respectively, $\mathfrak{Z}_t(\mathcal{C}^{(2)}) = \mathcal{C}$].

Now, let \mathcal{X} be a Banach \mathcal{C} -bimodule. Then $\mathcal{X}^{(1)}$ is also a Banach \mathcal{C} -bimodule by the following module actions:

$$\langle x^{(1)} \cdot c, x \rangle = \langle x^{(1)}, c \cdot x \rangle, \quad \langle c \cdot x^{(1)}, x \rangle = \langle x^{(1)}, x \cdot c \rangle,$$

where $c \in \mathcal{C}$, $x \in \mathcal{X}$, $x^{(1)} \in \mathcal{X}^{(1)}$. We may apply the same argument to show that for each $n \in \mathbb{N}$ $\mathcal{X}^{(n)}$ is a Banach \mathcal{C} -bimodule. A derivation from \mathcal{C} into \mathcal{X} is a continuous linear map $D : \mathcal{C} \rightarrow \mathcal{X}$ such that for every $c_1, c_2 \in \mathcal{C}$

$$D(c_1 c_2) = D(c_1) \cdot c_2 + c_1 \cdot D(c_2).$$

A derivation $D : \mathcal{C} \rightarrow \mathcal{C}^{(1)}$ is called cyclic if

$$\langle D(c_1), c_2 \rangle + \langle D(c_2), c_1 \rangle = 0 \quad (c_1, c_2 \in \mathcal{C}),$$

For $x \in \mathcal{X}$, define ad_x from \mathcal{C} into \mathcal{X} by $\text{ad}_x(c) = c \cdot x - x \cdot c$. It is easy to show that ad_x is a derivation; such derivations are called inner derivations. We denote the set of all derivations from \mathcal{C} into \mathcal{X} by $Z^1(\mathcal{C}, \mathcal{X})$. The first cohomology group $H^1(\mathcal{C}, \mathcal{X})$ is the quotient of the space of all derivations by the space of all inner derivations, and similarly the first-order cyclic cohomology group $H_\lambda^1(\mathcal{C}, \mathcal{X})$ is the quotient of the space of all cyclic derivations by the space of all inner derivations. In many situations triviality of these spaces are of considerable importance. In particular, \mathcal{C} is called contractible [respectively, amenable] if $H^1(\mathcal{C}, \mathcal{X}) = 0$ [respectively, $H^1(\mathcal{C}, \mathcal{X}^{(1)}) = 0$] for every Banach \mathcal{C} -bimodule \mathcal{X} and for $n \in \mathbb{N} \cup \{0\}$ it is called n -weakly amenable if $H^1(\mathcal{C}, \mathcal{C}^{(n)}) = 0$. Moreover, recall from [13] that \mathcal{C} is cyclic amenable if $H_\lambda^1(\mathcal{C}, \mathcal{C}^{(1)}) = 0$. For more information about these notions we refer the reader to the impressive references [5, 13].

3 Definition and some basic results

We commence this section by introducing the main object of study of this work. To this end, we need to recall some notations. Let \mathcal{A} and \mathfrak{A} be Banach algebras such that \mathcal{A} is a Banach \mathfrak{A} -bimodule, we say that the left and right actions of \mathfrak{A} on \mathcal{A} are compatible, if for each $a, b \in \mathcal{A}$ and $\beta \in \mathfrak{A}$,

$$\beta \cdot (ab) = (\beta \cdot a)b, \quad (ab) \cdot \beta = a(b \cdot \beta), \quad a(\beta \cdot b) = (a \cdot \beta)b.$$

and in the case where $\beta \cdot a = a \cdot \beta$, we say that \mathcal{A} is a symmetric \mathfrak{A} -bimodule.

Definition 3.1 Let \mathcal{A} and \mathfrak{A} be Banach algebras such that \mathcal{A} is a Banach \mathfrak{A} -bimodule with compatible actions. The amalgamated duplication of \mathfrak{A} along \mathcal{A} , denoted by $\mathcal{A} \rtimes \mathfrak{A}$, is defined as the Cartesian product $\mathcal{A} \times \mathfrak{A}$ with the algebra product

$$(a, \beta)(b, \gamma) = (ab + a \cdot \gamma + \beta \cdot b, \beta\gamma)$$

and with the norm $\|(a, \beta)\| = \|a\| + \|\beta\|$.

The following remark is now immediate:

Remark 3.2 Let \mathcal{A} and \mathfrak{A} be Banach algebras such that \mathcal{A} is a Banach \mathfrak{A} -bimodule with compatible actions. Then the following statements hold.

- (i) It is clear that $\mathcal{A} \rtimes \mathfrak{A}$ is a Banach algebra. Moreover, if we identify \mathcal{A} with $\mathcal{A} \times \{0\}$ and \mathfrak{A} with $\{0\} \times \mathfrak{A}$, then \mathcal{A} is a closed ideal of $\mathcal{A} \rtimes \mathfrak{A}$ and \mathfrak{A} is a closed subalgebra. Moreover

$$\mathcal{A} \rtimes \mathfrak{A} / \mathcal{A} \cong \mathfrak{A} \quad (\text{isometric isomorphism}).$$

This allows us to consider $\mathcal{A} \rtimes \mathfrak{A}$ as a strongly splitting Banach algebra extension of \mathfrak{A} by \mathcal{A} .

- (ii) $\mathcal{A} \rtimes \mathfrak{A}$ is commutative if and only if \mathcal{A} and \mathfrak{A} are commutative Banach algebra and \mathcal{A} is a symmetric \mathfrak{A} -bimodule.
- (iii) If \mathcal{A} and \mathfrak{A} are only commutative rings without Banach space structure, then $\mathcal{A} \rtimes \mathfrak{A}$ coincides with the amalgamated duplication of a ring \mathfrak{A} along a \mathfrak{A} -module \mathcal{A} , see [8, Page 3].

The following example shows that the amalgamated duplication of Banach algebras includes a large class of well-known Banach algebras.

Example 3.3 Let \mathfrak{A} be a Banach algebras. Observe that:

- (i) If \mathcal{A} is a nonzero Banach \mathfrak{A} -bimodule. Then we can consider \mathcal{A} as a Banach \mathfrak{A} -bimodule with compatible actions when \mathcal{A} is endowed with the zero algebra product. Hence, $\mathcal{A} \rtimes \mathfrak{A}$ is the module extension Banach algebra in the sense of [22]. Moreover, as has been discussed in [22], every triangular Banach algebra is isometrically isomorphic to a module extension Banach algebra. This allows us to consider triangular Banach algebras as an example of the amalgamated duplication of \mathfrak{A} along \mathcal{A} .
- (ii) If \mathcal{A} is a Banach algebras, $\sigma(\mathfrak{A}) \neq \emptyset$ and $\theta \in \sigma(\mathfrak{A})$. Then it is trivial that with the following actions

$$\beta \cdot a = a \cdot \beta = \theta(\beta)a \quad (a \in \mathcal{A}, \beta \in \mathfrak{A}),$$

\mathcal{A} is a Banach \mathfrak{A} -bimodule with compatible actions. Hence, in this case $\mathcal{A} \rtimes \mathfrak{A}$ is the θ -Lau product $\mathcal{A} \times_{\theta} \mathfrak{A}$ in the sense of [18].

Convention. For the rest of this paper we assume that \mathcal{A} and \mathfrak{A} are Banach algebras such that \mathcal{A} is a Banach \mathfrak{A} -bimodule with compatible actions, unless stated otherwise.

Let N be a left ideal of $\mathcal{A} \rtimes \mathfrak{A}$ and

$$I_N := \left\{ a \in \mathcal{A} : (a, \beta) \in N \text{ for some } \beta \in \mathfrak{A} \right\},$$

$$J_N := \left\{ \beta \in \mathfrak{A} : (a, \beta) \in N \text{ for some } a \in \mathcal{A} \right\}.$$

Then a routine computation shows that J_N is a left ideal of \mathfrak{A} and it is shown by examples (see [18, Example 2.8]) that in general we can neither expect I_N to be an ideal, nor to have $N = I_N \times J_N$.

The following result which is stated for left ideals is also true for right and two-sided ideals. Part (i) of this proposition is a generalization of two results proved by Monfared [18, Propositions 2.6 and 2.7], and in particular, parts (ii)-(vi) of this proposition can be considered as new results for Lau products of Banach algebras. Moreover, as far as we know the subject, this gives a characterization of the ideal structure of module extensions of Banach algebras and triangular Banach algebras.

Proposition 3.4 *If I and J are nonempty subspaces of \mathcal{A} and \mathfrak{A} , respectively, then the following assertions hold.*

- (i) *The subspace $I \times J$ is a left ideal of $\mathcal{A} \rtimes \mathfrak{A}$ if and only if I and J are left ideals of \mathcal{A} and \mathfrak{A} , respectively, and I is a left \mathfrak{A} -submodule of \mathcal{A} for which $\mathcal{A} \cdot J \subseteq I$.*
- (ii) *Let N be a left ideal of $\mathcal{A} \rtimes \mathfrak{A}$ containing $\{0\} \times \mathfrak{A}$. Then I_N is a left ideal of \mathcal{A} and $N = I_N \times \mathfrak{A}$.*
- (iii) *Let N be a left ideal of $\mathcal{A} \rtimes \mathfrak{A}$ containing $\mathcal{A} \times \{0\}$. Then J_N is a left ideal of \mathfrak{A} and $N = \mathcal{A} \times J_N$.*
- (iv) *The subspace I is a maximal left ideal and a left \mathfrak{A} -submodule of \mathcal{A} for which $\mathcal{A} \cdot \mathfrak{A} \subseteq I$ if and only if $I \times \mathfrak{A}$ is a maximal left ideal of $\mathcal{A} \rtimes \mathfrak{A}$.*
- (v) *The subspace J is a maximal left ideal of \mathfrak{A} if and only if $\mathcal{A} \times J$ is a maximal left ideal of $\mathcal{A} \rtimes \mathfrak{A}$.*
- (vi) *The subspace $I \times J$ is a maximal left ideal of $\mathcal{A} \rtimes \mathfrak{A}$ if and only if either $I = \mathcal{A}$ with J is a maximal left ideal of \mathfrak{A} or $J = \mathfrak{A}$ with I is both a maximal left ideal and a left \mathfrak{A} -submodule of \mathcal{A} satisfying $\mathcal{A} \cdot \mathfrak{A} \subseteq I$.*

Proof. For brevity we only give the proof for assertions (ii), (iv) and (vi).

(ii) Let N be a left ideal of $\mathcal{A} \rtimes \mathfrak{A}$ containing $\{0\} \times \mathfrak{A}$ and let $a \in I_N$. Then there is a β in \mathfrak{A} such that $(a, \beta) \in N$. It follows that $(a, 0) \in N$. From this, we can deduce that I_N is a left ideal of \mathcal{A} and $N = I_N \times \mathfrak{A}$.

(iv) Let I be a left \mathfrak{A} -submodule and a maximal left ideal of \mathcal{A} for which $\mathcal{A} \cdot \mathfrak{A} \subseteq I$. That $I \times \mathfrak{A}$ is a left ideal of $\mathcal{A} \rtimes \mathfrak{A}$ follows from part (i). Now, let N be any left ideal of $\mathcal{A} \rtimes \mathfrak{A}$ with

$$I \times \mathfrak{A} \subsetneq N \subseteq \mathcal{A} \rtimes \mathfrak{A}.$$

Then, by part (ii), I_N is a left ideal of \mathcal{A} and $N = I_N \rtimes \mathfrak{A}$. It follows that $I \subsetneq I_N \subseteq \mathcal{A}$ and thus $I_N = \mathcal{A}$. Hence $I \rtimes \mathfrak{A}$ is maximal.

Conversely, let $I \rtimes \mathfrak{A}$ be a maximal left ideal of $\mathcal{A} \rtimes \mathfrak{A}$ and let I' be any left ideal of \mathcal{A} for which $I \subsetneq I' \subseteq \mathcal{A}$. Then

$$I \rtimes \mathfrak{A} \subsetneq I' \rtimes \mathfrak{A} \subseteq \mathcal{A} \rtimes \mathfrak{A}.$$

As $I \rtimes \mathfrak{A}$ is maximal, $I' = \mathcal{A}$. Therefore I is a maximal left ideal of \mathfrak{A} .

(vi) Let $I \times J$ be a maximal left ideal of $\mathcal{A} \rtimes \mathfrak{A}$ and $I \subsetneq \mathcal{A}$. Then

$$I \times J \subsetneq \mathcal{A} \times J \subseteq \mathcal{A} \rtimes \mathfrak{A}.$$

Since $I \times J$ is a maximal left ideal of $\mathcal{A} \rtimes \mathfrak{A}$, we have $J = \mathfrak{A}$. Similarly, if $J \subsetneq \mathfrak{A}$, then $I = \mathcal{A}$. The converse follows from parts (iv) and (v). \blacksquare

Let \mathcal{C} be a Banach algebra and let \mathcal{X} and \mathcal{Y} be two Banach \mathcal{C} -bimodules. An operator $T : \mathcal{X} \rightarrow \mathcal{Y}$ is called a *left \mathcal{C} -module map* if $T(c \cdot x) = c \cdot T(x)$ for all $c \in \mathcal{C}$ and $x \in \mathcal{X}$. *Right \mathcal{C} -module* and *\mathcal{C} -bimodule maps* are defined similarly. We denote by $\text{Hom}_{\mathcal{C}}(\mathcal{X}, \mathcal{Y})$ the space of all bounded left \mathcal{C} -module maps from \mathcal{X} into \mathcal{Y} . Define $\text{LM}(\mathcal{C}) := \text{Hom}_{\mathcal{C}}(\mathcal{C}, \mathcal{C})$ to be the left multiplier algebra of \mathcal{C} .

The following result, stated for left multipliers, is also true for right multipliers. In particular, it can be considered as a new result for certain examples of $\mathcal{A} \rtimes \mathfrak{A}$ which were introduced in Example 3.3.

Proposition 3.5 *The operator T is in $\text{LM}(\mathcal{A} \rtimes \mathfrak{A})$ if and only if there exists some $T_{\mathcal{A}}^1 \in \text{Hom}_{\mathfrak{A}}(\mathcal{A}, \mathcal{A})$, $T_{\mathfrak{A}}^1 \in \text{Hom}_{\mathfrak{A}}(\mathfrak{A}, \mathcal{A})$, $T_{\mathfrak{A}}^2 \in \text{LM}(\mathfrak{A})$ and $T_{\mathcal{A}}^2 \in \text{Hom}_{\mathfrak{A}}(\mathcal{A}, \mathfrak{A})$ such that for each $a, b \in \mathcal{A}$ and $\beta \in \mathfrak{A}$*

$$(i) \quad T((a, \beta)) = (T_{\mathcal{A}}^1(a) + T_{\mathfrak{A}}^1(\beta), T_{\mathcal{A}}^2(a) + T_{\mathfrak{A}}^2(\beta)).$$

$$(ii) \quad T_{\mathcal{A}}^1(ab) = aT_{\mathcal{A}}^1(b) + a \cdot T_{\mathcal{A}}^2(b).$$

$$(iii) \quad T_{\mathcal{A}}^1(a \cdot \beta) = aT_{\mathfrak{A}}^1(\beta) + a \cdot T_{\mathfrak{A}}^2(\beta).$$

$$(iv) \quad T_{\mathcal{A}}^2(ab) = T_{\mathcal{A}}^2(a \cdot \beta) = 0.$$

Proof. We only need to prove the “if” part of this proposition, which is the essential part of it. Suppose that $T \in \text{LM}(\mathcal{A} \rtimes \mathfrak{A})$. Then, there exists bounded linear maps $T_1 : \mathcal{A} \rtimes \mathfrak{A} \rightarrow \mathcal{A}$ and $T_2 : \mathcal{A} \rtimes \mathfrak{A} \rightarrow \mathfrak{A}$ such that $T = (T_1, T_2)$. Let

$$T_{\mathcal{A}}^1(a) = T_1((a, 0)), \quad T_{\mathcal{A}}^2(a) = T_2((a, 0)) \quad (a \in \mathcal{A}),$$

and

$$T_{\mathfrak{A}}^1(\beta) = T_1((0, \beta)), \quad T_{\mathfrak{A}}^2(\beta) = T_2((0, \beta)) \quad (\beta \in \mathfrak{A}).$$

Then trivially $T_{\mathcal{A}}^1$, $T_{\mathfrak{A}}^1$, $T_{\mathcal{A}}^2$ and $T_{\mathfrak{A}}^2$ are linear maps satisfying (i). Moreover, for each $a, b \in \mathcal{A}$ and $\beta, \gamma \in \mathfrak{A}$, we observe

$$\begin{aligned} T((a, \beta)(b, \gamma)) &= T((ab + a \cdot \gamma + \beta \cdot b, \beta\gamma)) \\ &= (T_{\mathcal{A}}^1(ab + a \cdot \gamma + \beta \cdot b) + T_{\mathfrak{A}}^1(\beta\gamma), T_{\mathcal{A}}^2(ab + a \cdot \gamma + \beta \cdot b) + T_{\mathfrak{A}}^2(\beta\gamma)), \end{aligned} \quad (1)$$

and

$$\begin{aligned} (a, \beta)T((b, \gamma)) &= (a, \beta)\left(T_{\mathcal{A}}^1(b) + T_{\mathfrak{A}}^1(\gamma), T_{\mathcal{A}}^2(b) + T_{\mathfrak{A}}^2(\gamma)\right) \\ &= \left(aT_{\mathcal{A}}^1(b) + aT_{\mathfrak{A}}^1(\gamma) + a \cdot T_{\mathcal{A}}^2(b) + a \cdot T_{\mathfrak{A}}^2(\gamma) \right. \\ &\quad \left. + \beta \cdot T_{\mathcal{A}}^1(b) + \beta \cdot T_{\mathfrak{A}}^1(\gamma), \beta T_{\mathcal{A}}^2(b) + \beta T_{\mathfrak{A}}^2(\gamma)\right). \end{aligned} \quad (2)$$

Therefore, Equalities (1) and (2) imply that

$$\begin{aligned} T_{\mathcal{A}}^1(ab + a \cdot \gamma + \beta \cdot b) + T_{\mathfrak{A}}^1(\beta\gamma) &= aT_{\mathcal{A}}^1(b) + aT_{\mathfrak{A}}^1(\gamma) + a \cdot T_{\mathcal{A}}^2(b) \\ &\quad + a \cdot T_{\mathfrak{A}}^2(\gamma) + \beta \cdot T_{\mathcal{A}}^1(b) + \beta \cdot T_{\mathfrak{A}}^1(\gamma), \end{aligned} \quad (3)$$

and

$$T_{\mathcal{A}}^2(ab + a \cdot \gamma + \beta \cdot b) + T_{\mathfrak{A}}^2(\beta\gamma) = \beta T_{\mathcal{A}}^2(b) + \beta T_{\mathfrak{A}}^2(\gamma). \quad (4)$$

Applying (3) and (4) for suitable values of a, b, β, γ show that $T_{\mathcal{A}}^1 \in \text{Hom}_{\mathfrak{A}}(\mathcal{A}, \mathcal{A})$, $T_{\mathfrak{A}}^1 \in \text{Hom}_{\mathfrak{A}}(\mathfrak{A}, \mathcal{A})$, $T_{\mathfrak{A}}^2 \in \text{LM}(\mathfrak{A})$ and $T_{\mathcal{A}}^2 \in \text{Hom}_{\mathfrak{A}}(\mathcal{A}, \mathfrak{A})$ and the Equalities (ii)-(iv) are also satisfied. ■

From now on, for a set Y in a Banach space, $\langle Y \rangle$ denotes the closed linear span of Y in the space.

Corollary 3.6 *Suppose that either $\langle \mathcal{A}^2 \rangle = \mathcal{A}$ or $\langle \mathcal{A} \cdot \mathfrak{A} \rangle = \mathcal{A}$. Then $T \in \text{LM}(\mathcal{A} \rtimes \mathfrak{A})$ if and only if there exists some $T_{\mathcal{A}}^1 \in \text{Hom}_{\mathfrak{A}}(\mathcal{A}, \mathcal{A}) \cap \text{LM}(\mathcal{A})$, $T_{\mathfrak{A}}^1 \in \text{Hom}_{\mathfrak{A}}(\mathfrak{A}, \mathcal{A})$ and $T_{\mathfrak{A}}^2 \in \text{LM}(\mathfrak{A})$ such that $T_{\mathcal{A}}^1(a \cdot \beta) = aT_{\mathfrak{A}}^1(\beta) + a \cdot T_{\mathfrak{A}}^2(\beta)$ and*

$$T((a, \beta)) = (T_{\mathcal{A}}^1(a) + T_{\mathfrak{A}}^1(\beta), T_{\mathfrak{A}}^2(\beta))$$

for all $a \in \mathcal{A}$ and $\beta \in \mathfrak{A}$.

The first dual space of $\mathcal{A} \rtimes \mathfrak{A}$ can be identified with $\mathcal{A}^{(1)} \times \mathfrak{A}^{(1)}$ in the natural way

$$\langle (a^{(1)}, \beta^{(1)}), (a, \beta) \rangle = \langle a^{(1)}, a \rangle + \langle \beta^{(1)}, \beta \rangle,$$

where $a \in \mathcal{A}$, $\beta \in \mathfrak{A}$, $a^{(1)} \in \mathcal{A}^{(1)}$ and $\beta^{(1)} \in \mathfrak{A}^{(1)}$. The dual norm on $\mathcal{A}^{(1)} \times \mathfrak{A}^{(1)}$ is of course the maximum norm $\|(a^{(1)}, \beta^{(1)})\| = \max \{\|a^{(1)}\|, \|\beta^{(1)}\|\}$.

The proof of Proposition 3.8 below, which characterize the spectrum of the Banach algebra $\mathcal{A} \rtimes \mathfrak{A}$, relies on the following lemma.

Lemma 3.7 *If $\varphi \in \sigma(\mathcal{A})$, then there exists a unique linear functional $\tilde{\varphi}$ in $\sigma(\mathfrak{A}) \cup \{0\}$ such that*

$$\varphi(a \cdot \beta) = \varphi(\beta \cdot a) = \varphi(a)\tilde{\varphi}(\beta) \quad (a \in \mathcal{A} \text{ and } \beta \in \mathfrak{A}). \quad (5)$$

In particular, if either $\langle \mathcal{A} \cdot \mathfrak{A} \rangle = \mathcal{A}$ or $\langle \mathfrak{A} \cdot \mathcal{A} \rangle = \mathcal{A}$, then $\tilde{\varphi} \neq 0$.

Proof. Let $\varphi \in \sigma(\mathcal{A})$ and $a_0, a'_0 \in \mathcal{A}$ be such that $\varphi(a_0) = \varphi(a'_0) = 1$. The compatibility of the left and right actions of \mathfrak{A} on \mathcal{A} imply that for each $\beta \in \mathfrak{A}$,

$$\varphi(\beta \cdot a_0) = \varphi(a'_0)\varphi(\beta \cdot a_0) = \varphi((a'_0 \cdot \beta)a_0) = \varphi(a'_0 \cdot \beta) = \varphi(a'_0(\beta \cdot a'_0)) = \varphi(\beta \cdot a'_0).$$

We now, define $\tilde{\varphi} : \mathfrak{A} \rightarrow \mathbb{C}$ by

$$\tilde{\varphi}(\beta) := \varphi(\beta \cdot a_0) \quad (\beta \in \mathfrak{A}).$$

The preceding equation shows that the definition is independent of a_0 and hence $\tilde{\varphi}$ is a well-defined bounded linear functional on \mathfrak{A} . In particular, we have

$$\varphi(a_0 \cdot \beta) = \varphi((a_0 \cdot \beta)a_0) = \varphi(\beta \cdot a_0) \quad (\beta \in \mathfrak{A}).$$

It follows that

$$\varphi(\beta \cdot a) = \varphi(a_0 \cdot \beta)\varphi(a) = \varphi(\beta \cdot a_0)\varphi(a) = \tilde{\varphi}(\beta)\varphi(a),$$

and

$$\varphi(a \cdot \beta) = \varphi(a \cdot (\beta \cdot a_0)) = \varphi(a)\varphi(\beta \cdot a_0) = \tilde{\varphi}(\beta)\varphi(a),$$

where $a \in \mathcal{A}$ and $\beta \in \mathfrak{A}$. Finally, it should be noted that the uniqueness and multiplicativity of $\tilde{\varphi}$ follow similarly. \blacksquare

Note: (i) For the rest of this paper, we shall use the letters $\tilde{\varphi}$ exclusively to denote the unique multiplicative linear functional associated to $\varphi \in \sigma(\mathcal{A})$ which satisfies in (5).

(ii) If $\sigma(\mathcal{A}) \neq \emptyset$ and either $\langle \mathcal{A} \cdot \mathfrak{A} \rangle = \mathcal{A}$ or $\langle \mathfrak{A} \cdot \mathcal{A} \rangle = \mathcal{A}$, then $\sigma(\mathfrak{A}) \neq \emptyset$.

Proposition 3.8 *Let*

$$E := \{(\varphi, \tilde{\varphi}) : \varphi \in \sigma(\mathcal{A})\} \quad \text{and} \quad F := \{(0, \psi) : \psi \in \sigma(\mathfrak{A})\}.$$

Set $E = \emptyset$ [respectively, $F = \emptyset$] if $\sigma(\mathcal{A}) = \emptyset$ [respectively, $\sigma(\mathfrak{A}) = \emptyset$]. Then E and F are disjoint and $\sigma(\mathcal{A} \rtimes \mathfrak{A}) = E \cup F$.

Proof. It is clear that $E \cap F = \emptyset$ and $E \cup F \subseteq \sigma(\mathcal{A} \rtimes \mathfrak{A})$. In order to prove that $\sigma(\mathcal{A} \rtimes \mathfrak{A}) \subseteq E \cup F$, suppose that $(\varphi, \psi) \in \sigma(\mathcal{A} \rtimes \mathfrak{A})$ and $(a, \beta), (b, \gamma) \in \mathcal{A} \rtimes \mathfrak{A}$. Observe that

$$\langle (\varphi, \psi), (a, \beta)(b, \gamma) \rangle = \langle (\varphi, \psi), (a, \beta) \rangle \langle (\varphi, \psi), (b, \gamma) \rangle.$$

It follows that

$$\varphi(ab + a \cdot \gamma + \beta \cdot b) + \psi(\beta\gamma) = \varphi(a)\varphi(b) + \varphi(a)\psi(\gamma) + \psi(\beta)\varphi(b) + \psi(\beta)\psi(\gamma).$$

Taking $\beta = \gamma = 0$, we have $\varphi(ab) = \varphi(a)\varphi(b)$ and thus $\varphi \in \sigma(\mathcal{A}) \cup \{0\}$. Similarly, we can see that $\psi \in \sigma(\mathfrak{A}) \cup \{0\}$. Now, if $\beta = 0$ and $b = 0$, then $\varphi(a \cdot \gamma) = \varphi(a)\psi(\gamma)$, similarly $\varphi(\beta \cdot b) = \psi(\beta)\varphi(b)$. The equality $\varphi = 0$ implies that $\psi \neq 0$, and if $\varphi \neq 0$, then $\psi = \tilde{\varphi}$ by Lemma 3.7. \blacksquare

Next we turn our attention to the question of semisimplicity and regularity of $\mathcal{A} \rtimes \mathfrak{A}$.

Proposition 3.9 *The following statements hold.*

- (i) $\mathcal{A} \rtimes \mathfrak{A}$ is semisimple if and only if both \mathcal{A} and \mathfrak{A} are semisimple.
- (ii) $\mathcal{A} \rtimes \mathfrak{A}$ is regular if and only if both \mathcal{A} and \mathfrak{A} are regular.

Proof. (i) Let \mathcal{A} and \mathfrak{A} be semisimple. We show that the Gelfand homomorphism $\Gamma_{\mathcal{A} \rtimes \mathfrak{A}}$ is injective. To this end, suppose that $a \in \mathcal{A}$ and $\beta \in \mathfrak{A}$ such that $\widehat{(a, \beta)} = 0$. From this, by Proposition 3.8 and the equality $\widehat{(a, \beta)} = (\hat{a}, \hat{\beta})$, we deduced that

$$\begin{aligned} 0 &= \langle (\hat{a}, \hat{\beta}), (\varphi, \tilde{\varphi}) \rangle = \varphi(a) + \tilde{\varphi}(\beta) & (\varphi \in \sigma(\mathcal{A})), \\ 0 &= \langle (\hat{a}, \hat{\beta}), (0, \psi) \rangle = \psi(\beta) & (\psi \in \sigma(\mathfrak{A})). \end{aligned}$$

This implies that $a = \beta = 0$. The converse can be proved similarly.

(ii) Since $\mathcal{A} \times \{0\}$ is a closed ideal of $\mathcal{A} \rtimes \mathfrak{A}$ and $\mathcal{A} \rtimes \mathfrak{A} / \mathcal{A} \times \{0\}$ is isometrically isomorphic to \mathfrak{A} , it follows from [14, Theorems 4.2.6 and 4.3.8] that $\mathcal{A} \rtimes \mathfrak{A}$ is regular if and only if \mathcal{A} and \mathfrak{A} are regular. \blacksquare

Now, let $a, b \in \mathcal{A}$, $\beta, \gamma \in \mathfrak{A}$, $a^{(1)} \in \mathcal{A}^{(1)}$, $a^{(2)} \in \mathcal{A}^{(2)}$ and $\beta^{(2)} \in \mathfrak{A}^{(2)}$. Then we can extend the left and right actions of \mathfrak{A} on \mathcal{A} to compatible actions of $\mathfrak{A}^{(2)}$ on $\mathcal{A}^{(2)}$ via

$$\begin{aligned} \langle a^{(2)} \bullet \beta^{(2)}, a^{(1)} \rangle &= \langle a^{(2)}, \beta^{(2)} \bullet a^{(1)} \rangle, \\ \langle \beta^{(2)} \bullet a^{(1)}, a \rangle &= \langle \beta^{(2)}, a^{(1)} \bullet a \rangle, \\ \langle a^{(1)} \bullet a, \gamma \rangle &= \langle a^{(1)}, a \cdot \gamma \rangle, \end{aligned}$$

and

$$\begin{aligned} \langle \beta^{(2)} \bullet a^{(2)}, a^{(1)} \rangle &= \langle \beta^{(2)}, a^{(2)} \bullet a^{(1)} \rangle, \\ \langle a^{(2)} \bullet a^{(1)}, \beta \rangle &= \langle a^{(2)}, a^{(1)} \bullet \beta \rangle, \\ \langle a^{(1)} \bullet \beta, b \rangle &= \langle a^{(1)}, \beta \cdot b \rangle. \end{aligned}$$

Similarly, by using symmetry, we may consider $\mathcal{A}^{(2)}$ as a Banach $\mathfrak{A}^{(2)}$ -bimodule with compatible actions via the following module actions

$$\begin{aligned} \langle a^{(2)} \blacktriangle \beta^{(2)}, a^{(1)} \rangle &= \langle \beta^{(2)}, a^{(1)} \blacktriangle a^{(2)} \rangle, \\ \langle a^{(1)} \blacktriangle a^{(2)}, \beta \rangle &= \langle a^{(2)}, \beta \blacktriangle a^{(1)} \rangle, \\ \langle \beta \blacktriangle a^{(1)}, b \rangle &= \langle a^{(1)}, b \cdot \beta \rangle, \end{aligned}$$

and

$$\begin{aligned} \langle \beta^{(2)} \blacktriangle a^{(2)}, a^{(1)} \rangle &= \langle a^{(2)}, a^{(1)} \blacktriangle \beta^{(2)} \rangle, \\ \langle a^{(1)} \blacktriangle \beta^{(2)}, a \rangle &= \langle \beta^{(2)}, a \blacktriangle a^{(1)} \rangle, \\ \langle a \blacktriangle a^{(1)}, \gamma \rangle &= \langle a^{(1)}, \gamma \cdot a \rangle. \end{aligned}$$

Hence, one can calculate the first and second Arens products on $(\mathcal{A} \times \mathfrak{A})^{(2)}$ as follows

$$\begin{aligned} (b^{(1)}, \gamma^{(1)}) \circ (b, \gamma) &= (b^{(1)} \circ b + b^{(1)} \bullet \gamma, b^{(1)} \bullet b + \gamma^{(1)} \circ \gamma), \\ (b^{(2)}, \gamma^{(2)}) \circ (b^{(1)}, \gamma^{(1)}) &= (b^{(2)} \circ b^{(1)} + \gamma^{(2)} \bullet b^{(1)}, b^{(2)} \bullet b^{(1)} + \gamma^{(2)} \circ \gamma^{(1)}), \\ (a^{(2)}, \beta^{(2)}) \circ (b^{(2)}, \gamma^{(2)}) &= (a^{(2)} \circ b^{(2)} + a^{(2)} \bullet \gamma^{(2)} + \beta^{(2)} \bullet b^{(2)}, \beta^{(2)} \circ \gamma^{(2)}), \end{aligned}$$

and

$$\begin{aligned} (b, \gamma) \triangle (b^{(1)}, \gamma^{(1)}) &= (b \triangle b^{(1)} + \gamma \blacktriangle b^{(1)}, b \blacktriangle b^{(1)} + \gamma \triangle \gamma^{(1)}), \\ (b^{(1)}, \gamma^{(1)}) \triangle (a^{(2)}, \beta^{(2)}) &= (b^{(1)} \triangle a^{(2)} + b^{(1)} \blacktriangle \beta^{(2)}, b^{(1)} \blacktriangle a^{(2)} + \gamma^{(1)} \triangle \beta^{(2)}), \\ (a^{(2)}, \beta^{(2)}) \triangle (b^{(2)}, \gamma^{(2)}) &= (a^{(2)} \triangle b^{(2)} + a^{(2)} \blacktriangle \gamma^{(2)} + \beta^{(2)} \blacktriangle b^{(2)}, \beta^{(2)} \triangle \gamma^{(2)}), \end{aligned}$$

where $(b, \gamma) \in \mathcal{A} \rtimes \mathfrak{A}$, $(b^{(1)}, \gamma^{(1)}) \in (\mathcal{A} \rtimes \mathfrak{A})^{(1)}$ and $(a^{(2)}, \beta^{(2)}), (b^{(2)}, \gamma^{(2)}) \in (\mathcal{A} \rtimes \mathfrak{A})^{(2)}$. We summarize these observations in the next result which shows that the first and second Arens products defined on $(\mathcal{A} \rtimes \mathfrak{A})^{(2)}$ behaves in a natural way.

Proposition 3.10 *Suppose that $\mathcal{A}^{(2)}$, $\mathfrak{A}^{(2)}$, and $(\mathcal{A} \rtimes \mathfrak{A})^{(2)}$ are equipped with their first (resp. second) Arens products. Then $\mathcal{A}^{(2)}$ is a Banach $\mathfrak{A}^{(2)}$ -bimodule with compatible actions and in particular*

$$(\mathcal{A} \rtimes \mathfrak{A})^{(2)} \cong \mathcal{A}^{(2)} \rtimes \mathfrak{A}^{(2)},$$

where \cong denotes the isometric algebra isomorphism.

Let us recall from [10] that the topological centres of the left and right module actions of $\mathfrak{A}^{(2)}$ on $\mathcal{A}^{(2)}$ are as follows:

$$\mathfrak{Z}_{\mathfrak{A}}(\mathcal{A}^{(2)}) = \left\{ a^{(2)} \in \mathcal{A}^{(2)} : a^{(2)} \bullet \gamma^{(2)} = a^{(2)} \blacktriangle \gamma^{(2)} \text{ for all } \gamma^{(2)} \in \mathfrak{A}^{(2)} \right\},$$

and

$$\mathfrak{Z}_{\mathcal{A}}(\mathfrak{A}^{(2)}) = \left\{ \beta^{(2)} \in \mathfrak{A}^{(2)} : \beta^{(2)} \bullet b^{(2)} = \beta^{(2)} \blacktriangle b^{(2)} \text{ for all } b^{(2)} \in \mathcal{A}^{(2)} \right\}.$$

Moreover, the actions of \mathfrak{A} on \mathcal{A} are Arens regular if

$$\mathfrak{Z}_{\mathfrak{A}}(\mathcal{A}^{(2)}) = \mathcal{A}^{(2)}, \quad \text{and} \quad \mathfrak{Z}_{\mathcal{A}}(\mathfrak{A}^{(2)}) = \mathfrak{A}^{(2)}$$

and they are strongly Arens irregular if

$$\mathfrak{Z}_{\mathfrak{A}}(\mathcal{A}^{(2)}) = \mathcal{A}, \quad \text{and} \quad \mathfrak{Z}_{\mathcal{A}}(\mathfrak{A}^{(2)}) = \mathfrak{A}.$$

The following proposition is an immediate consequence of Proposition 3.10.

Proposition 3.11 *The following equality holds for the first topological centre of $\mathcal{A} \rtimes \mathfrak{A}$:*

$$\mathfrak{Z}_t((\mathcal{A} \rtimes \mathfrak{A})^{(2)}) = \left(\mathfrak{Z}_t(\mathcal{A}^{(2)}) \cap \mathfrak{Z}_{\mathfrak{A}}(\mathcal{A}^{(2)}) \right) \rtimes \left(\mathfrak{Z}_t(\mathfrak{A}^{(2)}) \cap \mathfrak{Z}_{\mathcal{A}}(\mathfrak{A}^{(2)}) \right).$$

In particular, $(\mathcal{A} \rtimes \mathfrak{A})^{(2)}$ is Arens regular [respectively, strongly Arens irregular] if and only if both \mathcal{A} and \mathfrak{A} are Arens regular [respectively, strongly Arens irregular] and \mathfrak{A} act regularly (resp. strongly irregular) on \mathcal{A} .

We end this section by noting that, from induction and the previous analysis of the first Arens product, one can show that the $(\mathcal{A} \rtimes \mathfrak{A})$ -bimodule actions on $(\mathcal{A} \rtimes \mathfrak{A})^{(n)}$ are formulated as follows:

$$(a, \beta) \triangle (a^{(n)}, \beta^{(n)}) = \begin{cases} (a \triangle a^{(n)} + \beta \blacktriangle a^{(n)} + a \blacktriangle \beta^{(n)}, \beta \triangle \beta^{(n)}) & \text{if } n \text{ is even} \\ (a \triangle a^{(n)} + \beta \blacktriangle a^{(n)}, a \blacktriangle a^{(n)} + \beta \triangle \beta^{(n)}) & \text{if } n \text{ is odd,} \end{cases}$$

and

$$(a^{(n)}, \beta^{(n)}) \circ (a, \beta) = \begin{cases} (a^{(n)} \circ a + a^{(n)} \bullet \beta + \beta^{(n)} \bullet a, \beta^{(n)} \circ \beta) & \text{if } n \text{ is even} \\ (a^{(n)} \circ a + a^{(n)} \bullet \beta, a^{(n)} \bullet a + \beta^{(n)} \circ \beta) & \text{if } n \text{ is odd} \end{cases}$$

where $(a, \beta) \in \mathcal{A} \times \mathfrak{A}$, $(a^{(n)}, \beta^{(n)}) \in \mathcal{A}^{(n)} \times \mathfrak{A}^{(n)}$ and $n \in \mathbb{Z}^+$.

4 Certain Derivations on $\mathcal{A} \rtimes \mathfrak{A}$

This section studies the sets $Z^1(\mathcal{A} \rtimes \mathfrak{A}, (\mathcal{A} \rtimes \mathfrak{A})^{(n)})$ and $H^1(\mathcal{A} \rtimes \mathfrak{A}, (\mathcal{A} \rtimes \mathfrak{A})^{(n)})$ for all $n \in \mathbb{N} \cup \{0\}$; in particular, the question when the continuous map $D : \mathcal{A} \rtimes \mathfrak{A} \rightarrow (\mathcal{A} \rtimes \mathfrak{A})^{(1)}$ is a cyclic derivation.

First, we note that an argument similar to the proof of Proposition 3.5 gives the following result which characterize the set of all derivations from $\mathcal{A} \rtimes \mathfrak{A}$ into $\mathcal{A} \rtimes \mathfrak{A}$ -bimodule $(\mathcal{A} \rtimes \mathfrak{A})^{(n)}$ for the case where n is a odd positive integer number. The details are omitted.

Proposition 4.1 *A bounded linear map $D : \mathcal{A} \rtimes \mathfrak{A} \rightarrow (\mathcal{A} \rtimes \mathfrak{A})^{(2n+1)}$ is a derivation if and only if there exists derivations $D_{\mathcal{A}}^1 : \mathcal{A} \rightarrow \mathcal{A}^{(2n+1)}$, $D_{\mathfrak{A}}^1 : \mathfrak{A} \rightarrow \mathcal{A}^{(2n+1)}$, $D_{\mathfrak{A}}^2 : \mathfrak{A} \rightarrow \mathfrak{A}^{(2n+1)}$ and a bounded linear map $D_{\mathcal{A}}^2 : \mathcal{A} \rightarrow \mathfrak{A}^{(2n+1)}$ such that*

- (i) $D((a, \beta)) = (D_{\mathcal{A}}^1(a) + D_{\mathfrak{A}}^1(\beta), D_{\mathcal{A}}^2(a) + D_{\mathfrak{A}}^2(\beta)),$
- (ii) $D_{\mathcal{A}}^1(\beta \cdot a) = D_{\mathfrak{A}}^1(\beta) \circ a + \beta \blacktriangle D_{\mathcal{A}}^1(a)$ and $D_{\mathcal{A}}^1(a \cdot \beta) = a \triangle D_{\mathfrak{A}}^1(\beta) + D_{\mathcal{A}}^1(a) \bullet \beta$
- (iii) $D_{\mathcal{A}}^2(\beta \cdot a) = D_{\mathfrak{A}}^1(\beta) \bullet a + \beta \triangle D_{\mathcal{A}}^2(a)$ and $D_{\mathcal{A}}^2(a \cdot \beta) = a \blacktriangle D_{\mathfrak{A}}^1(\beta) + D_{\mathcal{A}}^2(a) \circ \beta,$
- (iv) $D_{\mathcal{A}}^2(ab) = D_{\mathcal{A}}^1(a) \bullet b + a \blacktriangle D_{\mathcal{A}}^1(b),$

for all $a, b \in \mathcal{A}$ and $\beta \in \mathfrak{A}$.

Now we examine inner derivation from $\mathcal{A} \rtimes \mathfrak{A}$ into $(\mathcal{A} \rtimes \mathfrak{A})^{(2n+1)}$.

Proposition 4.2 *Let $D \in Z^1(\mathcal{A} \rtimes \mathfrak{A}, (\mathcal{A} \rtimes \mathfrak{A})^{(2n+1)})$ for some $n \in \mathbb{N}$, and let $D_{\mathcal{A}}^1$, $D_{\mathfrak{A}}^1$, $D_{\mathcal{A}}^2$ and $D_{\mathfrak{A}}^2$ associated to D as in Proposition 4.1. If $a^{(2n+1)} \in \mathcal{A}^{(2n+1)}$ and $\beta^{(2n+1)} \in \mathfrak{A}^{(2n+1)}$, then $D = \text{ad}_{(a^{(2n+1)}, \beta^{(2n+1)})}$ if and only if $D_{\mathcal{A}}^1 = \text{ad}_{a^{(2n+1)}}$, $D_{\mathfrak{A}}^2 = \text{ad}_{\beta^{(2n+1)}}$, $D_{\mathfrak{A}}^1 = \text{ad}_{a^{(2n+1)}}$ and*

$$D_{\mathcal{A}}^2(a) = a \blacktriangle a^{(2n+1)} - a^{(2n+1)} \bullet a$$

for all $a \in \mathcal{A}$.

Proof. For the proof, we need to note only that if $D = \text{ad}_{(a^{(2n+1)}, \beta^{(2n+1)})}$ for some $a^{(2n+1)} \in \mathcal{A}^{(2n+1)}$ and $\beta^{(2n+1)} \in \mathfrak{A}^{(2n+1)}$, then

$$\begin{aligned} (D_{\mathcal{A}}^1(a), D_{\mathcal{A}}^2(a)) &= D((a, 0)) \\ &= \text{ad}_{(a^{(2n+1)}, \beta^{(2n+1)})}(a, 0) \\ &= (a, 0) \triangle (a^{(2n+1)}, \beta^{(2n+1)}) - (a^{(2n+1)}, \beta^{(2n+1)}) \circ (a, 0) \\ &= (a \triangle a^{(2n+1)}, a \blacktriangle a^{(2n+1)}) - (a^{(2n+1)} \circ a, a^{(2n+1)} \bullet a), \end{aligned}$$

for all $a \in \mathcal{A}$. It follows that

$$D_{\mathcal{A}}^1(a) = \text{ad}_{a^{(2n+1)}}(a) \quad \text{and} \quad D_{\mathcal{A}}^2(a) = a \blacktriangle a^{(2n+1)} - a^{(2n+1)} \bullet a \quad (a \in \mathcal{A}).$$

Similarly,

$$D_{\mathfrak{A}}^2(\beta) = \text{ad}_{\beta^{(2n+1)}}(b) \quad \text{and} \quad D_{\mathfrak{A}}^1(\beta) = \beta \blacktriangle a^{(2n+1)} - a^{(2n+1)} \bullet \beta \quad (\beta \in \mathfrak{A}),$$

and this completes the proof. \blacksquare

The following corollary is an immediate consequence of Propositions 4.1 and 4.2 above.

Corollary 4.3 *Let $n \in \mathbb{N}$ and $T \in \text{Hom}_{\mathfrak{A}}(\mathcal{A}, \mathfrak{A}^{(2n+1)})$ be such that $T|_{\mathcal{A}^2} = 0$. Let also D_T from $\mathcal{A} \rtimes \mathfrak{A}$ into $(\mathcal{A} \rtimes \mathfrak{A})^{(2n+1)}$ be defined by $D_T((a, \beta)) = (0, T(a))$. Then D_T is in $Z^1(\mathcal{A} \rtimes \mathfrak{A}, (\mathcal{A} \rtimes \mathfrak{A})^{(2n+1)})$. In particular, D_T is inner if and only if there exists $a^{(2n+1)} \in \mathcal{A}^{(2n+1)}$ such that $a \triangle a^{(2n+1)} = a^{(2n+1)} \circ a$, $\beta \blacktriangle a^{(2n+1)} = a^{(2n+1)} \bullet \beta$ and*

$$T(a) = a \blacktriangle a^{(2n+1)} - a^{(2n+1)} \bullet a,$$

for all $a \in \mathcal{A}$ and $\beta \in \mathfrak{A}$.

Also, Proposition 4.1 paves the way for characterizing the cyclic derivations on $\mathcal{A} \rtimes \mathfrak{A}$ as follows:

Proposition 4.4 *A continuous linear map $D : \mathcal{A} \rtimes \mathfrak{A} \rightarrow (\mathcal{A} \rtimes \mathfrak{A})^{(1)}$ is a cyclic derivation if and only if there exists cyclic derivations $D_{\mathcal{A}}^1 \in Z^1(\mathcal{A}, \mathcal{A}^{(1)})$ and $D_{\mathfrak{A}}^2 \in Z^1(\mathfrak{A}, \mathfrak{A}^{(1)})$, a derivation $D_{\mathfrak{A}}^1 \in Z^1(\mathfrak{A}, \mathcal{A}^{(1)})$ and a bounded linear map $D_{\mathcal{A}}^2 : \mathcal{A} \rightarrow \mathfrak{A}^{(1)}$ such that for each $a, b \in \mathcal{A}$ and $\beta \in \mathfrak{A}$*

- (i) $D((a, \beta)) = (D_{\mathcal{A}}^1(a) + D_{\mathfrak{A}}^1(\beta), D_{\mathcal{A}}^2(a) + D_{\mathfrak{A}}^2(\beta)),$
- (ii) $D_{\mathcal{A}}^1(\beta \cdot a) = D_{\mathfrak{A}}^1(\beta) \circ a + \beta \blacktriangle D_{\mathcal{A}}^1(a)$ and $D_{\mathcal{A}}^1(a \cdot \beta) = a \triangle D_{\mathfrak{A}}^1(\beta) + D_{\mathcal{A}}^1(a) \bullet \beta,$
- (iii) $D_{\mathcal{A}}^2(\beta \cdot a) = D_{\mathfrak{A}}^1(\beta) \bullet a + \beta \triangle D_{\mathcal{A}}^2(a)$ and $D_{\mathcal{A}}^2(a \cdot \beta) = a \blacktriangle D_{\mathfrak{A}}^1(\beta) + D_{\mathcal{A}}^2(a) \circ \beta,$
- (iv) $D_{\mathcal{A}}^2(ab) = D_{\mathcal{A}}^1(a) \bullet b + a \blacktriangle D_{\mathcal{A}}^1(b),$
- (v) $(D_{\mathcal{A}}^2)^* \circ \Phi + D_{\mathfrak{A}}^1 = 0$, where Φ is the usual injection from \mathfrak{A} into $\mathfrak{A}^{(2)}$ and $(D_{\mathcal{A}}^2)^*$ is the adjoint of the operator $D_{\mathcal{A}}^2$.

Proof. First assume that D is a cyclic derivation from $\mathcal{A} \rtimes \mathfrak{A}$ into $(\mathcal{A} \rtimes \mathfrak{A})^{(1)}$. By Proposition 4.1 there exists $D_{\mathcal{A}}^1 \in Z^1(\mathcal{A}, \mathcal{A}^{(1)})$, $D_{\mathfrak{A}}^2 \in Z^1(\mathfrak{A}, \mathfrak{A}^{(1)})$, $D_{\mathfrak{A}}^1 \in Z^1(\mathfrak{A}, \mathcal{A}^{(1)})$ and a bounded linear map $D_{\mathcal{A}}^2 : \mathcal{A} \rightarrow \mathfrak{A}^{(1)}$ satisfying the conditions (i)-(iv). Recall from the proof of Proposition 4.1 that if $D = (D_1, D_2)$, then

$$D_{\mathcal{A}}^1(a) = D_1((a, 0)), \quad D_{\mathcal{A}}^2(a) = D_2((a, 0)) \quad (a \in \mathcal{A}),$$

and

$$D_{\mathfrak{A}}^1(b) = D_1((0, \beta)), \quad D_{\mathfrak{A}}^2(\beta) = D_2((0, \beta)) \quad (\beta \in \mathfrak{A}).$$

Moreover, for each $(a, \beta), (b, \gamma)$ in $\mathcal{A} \rtimes \mathfrak{A}$, we have

$$\langle D((a, \beta)), (b, \gamma) \rangle + \langle D((b, \gamma)), (a, \beta) \rangle = 0. \quad (6)$$

It follows that

$$\langle D_{\mathcal{A}}^1(a), b \rangle + \langle D_{\mathcal{A}}^1(b), a \rangle = \langle D((a, 0)), (b, 0) \rangle + \langle D((b, 0)), (a, 0) \rangle = 0, \quad (7)$$

and

$$\langle D_{\mathfrak{A}}^2(\beta), \gamma \rangle + \langle D_{\mathfrak{A}}^2(\gamma), \beta \rangle = \langle D((0, \beta)), (0, \gamma) \rangle + \langle D((0, \gamma)), (0, \beta) \rangle = 0. \quad (8)$$

Hence $D_{\mathcal{A}}^1$ and $D_{\mathfrak{A}}^2$ are cyclic derivations. Now, by equalities (6)-(8), we deduce that

$$\langle D_{\mathcal{A}}^2(a), \gamma \rangle + \langle D_{\mathcal{A}}^2(b), \beta \rangle + \langle D_{\mathfrak{A}}^1(\beta), b \rangle + \langle D_{\mathfrak{A}}^1(\gamma), a \rangle = 0. \quad (9)$$

Choosing $\beta = 0$ in (9), we see that $(D_{\mathcal{A}}^2)^* \circ \Phi + D_{\mathfrak{A}}^1 = 0$. Finally, we note that the proof of the converse is not difficult and is omitted. \blacksquare

The following proposition is now immediate:

Proposition 4.5 *Let $D \in Z^1(\mathcal{A} \rtimes \mathfrak{A}, (\mathcal{A} \rtimes \mathfrak{A})^{(1)})$ be cyclic, and let $D_{\mathcal{A}}^1, D_{\mathfrak{A}}^1, D_{\mathcal{A}}^2$ and $D_{\mathfrak{A}}^2$ associated to D as in Proposition 4.4. If $a^{(1)} \in \mathcal{A}^{(1)}$ and $\beta^{(1)} \in \mathfrak{A}^{(1)}$, then $D = \text{ad}_{(a^{(1)}, \beta^{(1)})}$ if and only if $D_{\mathcal{A}}^1 = \text{ad}_{a^{(1)}}$, $D_{\mathfrak{A}}^2 = \text{ad}_{\beta^{(1)}}$, $D_{\mathfrak{A}}^1 = \text{ad}_{a^{(1)}}$ and $D_{\mathcal{A}}^2(a) = a \blacktriangle a^{(1)} - a^{(1)} \bullet a$ for all $a \in \mathcal{A}$.*

By an argument similar to the proof of Propositions 4.1 and 4.2 one can obtain the following result which gives a characterization for the set of all derivation from $\mathcal{A} \times \mathfrak{A}$ into $(\mathcal{A} \times \mathfrak{A})^{(2n)}$. The details are omitted.

Proposition 4.6 *A bounded linear map $D : \mathcal{A} \rtimes \mathfrak{A} \rightarrow (\mathcal{A} \rtimes \mathfrak{A})^{(2n)}$ is a derivation if and only if there exists derivations $D_{\mathfrak{A}}^2 : \mathfrak{A} \rightarrow \mathfrak{A}^{(2n)}$, $D_{\mathcal{A}}^1 : \mathcal{A} \rightarrow \mathcal{A}^{(2n)}$, $D_{\mathcal{A}}^2 \in B_{\mathfrak{A}}(\mathcal{A}, \mathfrak{A}^{(2n)})$ and a bounded linear map $D_{\mathcal{A}}^1 : \mathcal{A} \rightarrow \mathcal{A}^{(2n)}$ such that*

- (i) $D((a, \beta)) = (D_{\mathcal{A}}^1(a) + D_{\mathfrak{A}}^1(\beta), D_{\mathcal{A}}^2(a) + D_{\mathfrak{A}}^2(\beta)),$
- (ii) $D_{\mathcal{A}}^1(ab) = D_{\mathcal{A}}^1(a) \circ b + D_{\mathcal{A}}^2(a) \bullet b + a \triangle D_{\mathcal{A}}^1(b) + a \blacktriangle D_{\mathcal{A}}^2(b),$
- (iii) $D_{\mathcal{A}}^1(\beta \cdot a) = D_{\mathfrak{A}}^1(\beta) \circ a + D_{\mathfrak{A}}^2(\beta) \bullet a + \beta \blacktriangle D_{\mathcal{A}}^1(a),$
- (iv) $D_{\mathcal{A}}^1(a \cdot \beta) = a \triangle D_{\mathfrak{A}}^1(\beta) + a \blacktriangle D_{\mathfrak{A}}^2(\beta) + D_{\mathcal{A}}^1(a) \bullet \beta,$
- (v) $D_{\mathcal{A}}^2(ab) = 0,$

for all $a, b \in \mathcal{A}$ and $\beta, \gamma \in \mathfrak{A}$. Moreover, if $a^{(2n)} \in \mathcal{A}^{(2n)}$ and $\beta^{(2n)} \in \mathfrak{A}^{(2n)}$, then $D = \text{ad}_{(a^{(2n)}, \beta^{(2n)})}$ if and only if $D_{\mathfrak{A}}^1 = \text{ad}_{a^{(2n)}}$, $D_{\mathfrak{A}}^2 = \text{ad}_{\beta^{(2n)}}$, $D_{\mathcal{A}}^2 = 0$ and

$$D_{\mathcal{A}}^1(a) = \text{ad}_{a^{(2n)}}(a) + a \blacktriangle \beta^{(2n)} - \beta^{(2n)} \bullet a,$$

for all $a \in \mathcal{A}$.

The following two corollaries are immediate consequences of Proposition 4.6 above. Recall that, for each $n \in \mathbb{N} \cup \{0\}$, a bounded linear map $D : \mathcal{A} \rightarrow \mathcal{A}^{(2n)}$ is called a \mathfrak{A} -module derivation if

$$D(ab) = D(a) \circ b + a \triangle D(b) \quad (a, b \in \mathcal{A}),$$

and

$$D(\beta \cdot a) = \beta \blacktriangle D(a), \quad D(a \cdot \beta) = D(a) \bullet \beta \quad (\beta \in \mathfrak{A}, a \in \mathcal{A}).$$

The notation $Z_{\mathfrak{A}}^1(\mathcal{A}, \mathcal{A}^{(2n)})$ is used to denote the set of all module derivations from \mathcal{A} into $\mathcal{A}^{(2n)}$.

Corollary 4.7 *Suppose that $T \in Z_{\mathfrak{A}}^1(\mathcal{A}, \mathcal{A}^{(2n)})$ for some positive integer n . Then the bounded linear map D_T defined by*

$$D_T : \mathcal{A} \rtimes \mathfrak{A} \rightarrow (\mathcal{A} \rtimes \mathfrak{A})^{(2n)}; \quad D_T((a, \beta)) = (T(a), 0),$$

is in $Z^1(\mathcal{A} \rtimes \mathfrak{A}, (\mathcal{A} \rtimes \mathfrak{A})^{(2n)})$. Moreover, D_T is inner if and only if there exists $a^{(2n)} \in \mathcal{A}^{(2n)}$ such that $\beta \blacktriangle a^{(2n)} = a^{(2n)} \bullet \beta$ and $T(a) = \text{ad}_{a^{(2n)}}(a)$ for all $a \in \mathcal{A}$ and $\beta \in \mathfrak{A}$.

Corollary 4.8 *Suppose that $D_{\mathcal{A}} \in Z^1(\mathcal{A}, \mathcal{A}^{(2n)})$ and $T \in Z_{\mathfrak{A}}^1(\mathcal{A}, \mathcal{A}^{(2n)})$ for some positive integer n . Then $D_T : \mathcal{A} \rtimes \mathcal{A} \rightarrow (\mathcal{A} \rtimes \mathcal{A})^{(2n)}$ defined by*

$$D_T((a, b)) = (T(a), D_{\mathcal{A}}(b))$$

is in $Z^1(\mathcal{A} \rtimes \mathcal{A}, (\mathcal{A} \rtimes \mathcal{A})^{(2n)})$. Moreover, D_T is inner if and only if there exists $a^{(2n)} \in \mathcal{A}^{(2n)}$ such that $\beta \blacktriangle a^{(2n)} = a^{(2n)} \bullet \beta$ and $T(a) = \text{ad}_{a^{(2n)}}(a)$ for all $a \in \mathcal{A}$ and $\beta \in \mathfrak{A}$.

Finally, the results of this section lead to the following corollary which characterize the sets $Z^1(\mathcal{A} \rtimes \mathfrak{A}, (\mathcal{A} \rtimes \mathfrak{A})^{(n)})$ and $H^1(\mathcal{A} \rtimes \mathfrak{A}, (\mathcal{A} \rtimes \mathfrak{A})^{(n)})$ in the case where $n \in \mathbb{N} \cup \{0\}$ and \mathcal{A} is a unital Banach algebra.

Corollary 4.9 *Let \mathcal{A} be unital with the identity e . Then the following statements hold.*

(i) *Every derivation $D : \mathcal{A} \rtimes \mathfrak{A} \rightarrow (\mathcal{A} \rtimes \mathfrak{A})^{(2n+1)}$ is in the form of*

$$D((a, \beta)) = (D_{\mathcal{A}}^1(a) + D_{\mathfrak{A}}^1(\beta), D_{\mathcal{A}}^2(a) + D_{\mathfrak{A}}^2(\beta)) \quad (a \in \mathcal{A}, \beta \in \mathfrak{A}),$$

where $D_{\mathcal{A}}^1 : \mathcal{A} \rightarrow \mathcal{A}^{(2n+1)}$, $D_{\mathfrak{A}}^2 : \mathfrak{A} \rightarrow \mathfrak{A}^{(2n+1)}$, $D_{\mathfrak{A}}^1 : \mathfrak{A} \rightarrow \mathcal{A}^{(2n+1)}$ are derivations and $D_{\mathcal{A}}^2 : \mathcal{A} \rightarrow \mathfrak{A}^{(2n+1)}$ is a bounded linear map satisfying

$$D_{\mathfrak{A}}^1(\beta) = D_{\mathcal{A}}^1(e \cdot \beta) = D_{\mathcal{A}}^1(\beta \cdot e) \quad \text{and} \quad D_{\mathcal{A}}^2(a) = D_{\mathcal{A}}^1(a) \bullet e = e \blacktriangle D_{\mathcal{A}}^1(a).$$

(ii) *Every derivation $D : \mathcal{A} \rtimes \mathfrak{A} \rightarrow (\mathcal{A} \rtimes \mathfrak{A})^{(2n)}$ is in the form of*

$$D((a, \beta)) = (D_{\mathcal{A}}^1(a) + D_{\mathfrak{A}}^1(\beta), D_{\mathfrak{A}}^2(\beta)) \quad (a \in \mathcal{A}, \beta \in \mathfrak{A}),$$

where $D_{\mathcal{A}}^1 : \mathcal{A} \rightarrow \mathcal{A}^{(2n)}$, $D_{\mathfrak{A}}^2 : \mathfrak{A} \rightarrow \mathfrak{A}^{(2n)}$, $D_{\mathfrak{A}}^1 : \mathfrak{A} \rightarrow \mathcal{A}^{(2n+1)}$ are derivations satisfying

$$D_{\mathfrak{A}}^1(\beta) = D_{\mathcal{A}}^1(e \cdot \beta) - D_{\mathfrak{A}}^2(\beta) \bullet e = D_{\mathcal{A}}^1(\beta \cdot e) - e \blacktriangle D_{\mathfrak{A}}^2(\beta).$$

5 n-weak amenability and cyclic amenability

Our aim in this section is to investigate some notions of amenability for $\mathcal{A} \rtimes \mathfrak{A}$ in relation to the corresponding ones of \mathcal{A} and \mathfrak{A} . We commence with the following result which gives a sufficient condition for $(2n+1)$ -weak amenability of $\mathcal{A} \rtimes \mathfrak{A}$.

Proposition 5.1 *Suppose that n is a positive integer number for which $\langle \mathcal{A} \triangle \mathcal{A}^{(2n)} \rangle = \mathcal{A}^{(2n)}$ or $\langle \mathcal{A}^{(2n)} \circ \mathcal{A} \rangle = \mathcal{A}^{(2n)}$. If \mathcal{A} and \mathfrak{A} are $(2n+1)$ -weakly amenable, then $\mathcal{A} \rtimes \mathfrak{A}$ is $(2n+1)$ -weakly amenable.*

Proof. Suppose that $D \in Z^1(\mathcal{A} \rtimes \mathfrak{A}, (\mathcal{A} \rtimes \mathfrak{A})^{(2n+1)})$. Then, there exists $D_{\mathcal{A}}^1 \in Z^1(\mathcal{A}, \mathcal{A}^{(2n+1)})$, $D_{\mathfrak{A}}^1 \in Z^1(\mathfrak{A}, \mathcal{A}^{(2n+1)})$, $D_{\mathfrak{A}}^2 \in Z^1(\mathfrak{A}, \mathfrak{A}^{(2n+1)})$ and a bounded linear map $D_{\mathcal{A}}^2 \in Z^1(\mathcal{A}, \mathfrak{A}^{(2n+1)})$ such that

$$D((a, \beta)) = (D_{\mathcal{A}}^1(a) + D_{\mathfrak{A}}^1(\beta), D_{\mathcal{A}}^2(a) + D_{\mathfrak{A}}^2(\beta)),$$

for all $a \in \mathcal{A}$ and $\beta \in \mathfrak{A}$. By assumption $D_{\mathcal{A}}^1 = \text{ad}_{a^{(2n+1)}}$ and $D_{\mathfrak{A}}^2 = \text{ad}_{\beta^{(2n+1)}}$ for some $a^{(2n+1)} \in \mathcal{A}^{(2n+1)}$ and $\beta^{(2n+1)} \in \mathfrak{A}^{(2n+1)}$. From this, by Proposition 4.1, we have

$$\begin{aligned} D_{\mathcal{A}}^2(ab) &= \text{ad}_{a^{(2n+1)}}(a) \bullet b + a \blacktriangle \text{ad}_{a^{(2n+1)}}(b) \\ &= (a \triangle a^{(2n+1)} - a^{(2n+1)} \circ a) \bullet b + a \blacktriangle (b \triangle a^{(2n+1)} - a^{(2n+1)} \circ b) \\ &= a \blacktriangle (b \triangle a^{(2n+1)}) - (a^{(2n+1)} \circ a) \bullet b \\ &= (ab) \blacktriangle a^{(2n+1)} - a^{(2n+1)} \bullet (ab), \end{aligned}$$

for all $a, b \in \mathcal{A}$. Moreover, $(2n+1)$ -weak amenability of \mathcal{A} implies that $\langle \mathcal{A}^2 \rangle = \mathcal{A}$. Therefore,

$$D_{\mathcal{A}}^2(a) = a \blacktriangle a^{(2n+1)} - a^{(2n+1)} \bullet a,$$

On the other hand, by Propositions 4.1 and 4.2, we have

$$\text{ad}_{a^{(2n+1)}}(\beta \cdot a) = D_{\mathfrak{A}}^1(\beta) \circ a + \beta \blacktriangle \text{ad}_{a^{(2n+1)}}(a).$$

It follows that

$$\left(D_{\mathfrak{A}}^1(\beta) - (\beta \blacktriangle a^{(2n+1)} - a^{(2n+1)} \bullet \beta) \right) \circ a = 0.$$

We now invoke the equality $\langle \mathcal{A} \triangle \mathcal{A}^{(2n)} \rangle = \mathcal{A}^{(2n)}$ to conclude that

$$D_{\mathfrak{A}}^1(\beta) = \beta \blacktriangle a^{(2n+1)} - a^{(2n+1)} \bullet \beta.$$

Similarly, if $\langle \mathcal{A}^{(2n)} \circ \mathcal{A} \rangle = \mathcal{A}^{(2n)}$, using the equality

$$\text{ad}_{a^{(2n+1)}}(a \cdot \beta) = a \triangle D_{\mathfrak{A}}^1(\beta) + \text{ad}_{a^{(2n+1)}}(a) \bullet \beta,$$

we can show that the equality $D_{\mathfrak{A}}^1(\beta) = \beta \blacktriangle a^{(2n+1)} - a^{(2n+1)} \bullet \beta$ holds. Hence, $D = \text{ad}_{(a^{(2n+1)}, \beta^{(2n+1)})}$ by Proposition 4.2. \blacksquare

There does not seem to be an easy way to show that if $\mathcal{A} \rtimes \mathfrak{A}$ is $(2n+1)$ -weakly amenable, then \mathcal{A} is also. We believe that more information about the properties of \mathcal{A} , \mathfrak{A} and the actions of \mathfrak{A} on \mathcal{A} are needed for further conclusions. Hence, motivated by Proposition 4.1, it seems valuable to define the following property for the pair $(\mathcal{A}, \mathfrak{A})$.

Definition 5.2 We say that the pair $(\mathcal{A}, \mathfrak{A})$ has the property (\mathbb{H}_{2n+1}) if for each $D_{\mathcal{A}}^1 \in Z^1(\mathcal{A}, \mathcal{A}^{(2n+1)})$, there are $D_{\mathfrak{A}}^1 \in Z^1(\mathfrak{A}, \mathcal{A}^{(2n+1)})$ and a bounded linear map $D_{\mathcal{A}}^2 : \mathcal{A} \rightarrow \mathfrak{A}^{(2n+1)}$ satisfying the conditions (ii)-(iv) of Proposition 4.1.

Actually the class of the pair $(\mathcal{A}, \mathfrak{A})$ satisfying this property is quite rich. It contains for instance all the pairs $(\mathcal{A}, \mathfrak{A})$ such that

- (i) $\mathcal{A} = \mathfrak{A}$ and the compatible actions is the natural actions;
- (ii) \mathcal{A} is unital;
- (iii) \mathcal{A} has a bounded approximate identity and \mathcal{A} is an essential Banach \mathfrak{A} -bimodule;
- (iv) the compatible actions of \mathfrak{A} on \mathcal{A} are defined as follows:

$$\beta \cdot a = a \cdot \beta = \theta(\beta)a \quad (a \in \mathcal{A}, \beta \in \mathfrak{A}),$$

where $\theta \in \sigma(\mathfrak{A})$;

- (v) the compatible actions of \mathfrak{A} on \mathcal{A} are defined as follows:

$$\beta \cdot a = T(\beta)a, \quad a \cdot \beta = aT(\beta) \quad (a \in \mathcal{A}, \beta \in \mathfrak{A}),$$

where T is a norm decreasing homomorphism from \mathfrak{A} into \mathcal{A} .

Theorem 5.3 *If $\mathcal{A} \rtimes \mathfrak{A}$ is $(2n+1)$ -weakly amenable for some $n \in \mathbb{N} \cup \{0\}$, then the following assertions hold.*

- (i) \mathfrak{A} is $(2n+1)$ -weakly amenable.
- (ii) If $(\mathcal{A}, \mathfrak{A})$ has the property (\mathbb{H}_{2n+1}) , then \mathcal{A} is $(2n+1)$ -weakly amenable.

Proof. (i) Suppose that $D_{\mathfrak{A}}^2 \in Z^1(\mathfrak{A}, \mathfrak{A}^{(2n+1)})$. Then, by Proposition 4.1, the map $D : \mathcal{A} \rtimes \mathfrak{A} \rightarrow (\mathcal{A} \rtimes \mathfrak{A})^{(2n+1)}$ defined by

$$D((a, \beta)) = (0, D_{\mathfrak{A}}^2(\beta)) \quad ((a, \beta) \in \mathcal{A} \rtimes \mathfrak{A})$$

is a derivation and therefore it is inner by $(2n+1)$ -weak amenability of $\mathcal{A} \rtimes \mathfrak{A}$. We now invoke Proposition 4.2 to conclude that $D_{\mathfrak{A}}^2$ is inner.

(ii) Suppose that $D_{\mathcal{A}}^1 \in Z^1(\mathcal{A}, \mathcal{A}^{(2n+1)})$. Since the pair $(\mathcal{A}, \mathfrak{A})$ has the property \mathbb{H}_{2n+1} , there exists $D_{\mathfrak{A}}^1 \in Z^1(\mathfrak{A}, \mathcal{A}^{(2n+1)})$ and a bounded linear map $D_{\mathcal{A}}^2 : \mathcal{A} \rightarrow \mathfrak{A}^{(2n+1)}$ satisfying the conditions (ii)-(iv) of Proposition 4.1. From this, by Proposition 4.1, we can conclude that the map $D : \mathcal{A} \rtimes \mathfrak{A} \rightarrow (\mathcal{A} \rtimes \mathfrak{A})^{(2n+1)}$ defined by

$$D((a, \beta)) = (D_{\mathcal{A}}^1(a) + D_{\mathfrak{A}}^1(\beta), D_{\mathcal{A}}^2(a)) \quad ((a, \beta) \in \mathcal{A} \rtimes \mathfrak{A})$$

is a derivation and therefore it is inner by $(2n+1)$ -weak amenability of $\mathcal{A} \rtimes \mathfrak{A}$. This together with Proposition 4.2 implies that $D_{\mathcal{A}}^1$ is also inner and this completes the proof. \blacksquare

The next result gives a necessary and sufficient condition for n -weak amenability of $\mathcal{A} \rtimes \mathfrak{A}$ in relation to that of the n -weak amenability of \mathcal{A} and \mathfrak{A} for the case where \mathcal{A} is unital.

Theorem 5.4 *Let \mathcal{A} be unital and let $n \in \mathbb{N} \cup \{0\}$. Then $\mathcal{A} \rtimes \mathfrak{A}$ is n -weakly amenable if and only if \mathcal{A} and \mathfrak{A} are n -weakly amenable.*

Proof. Let $D : \mathcal{A} \rtimes \mathfrak{A} \rightarrow (\mathcal{A} \rtimes \mathfrak{A})^{(2n+1)}$ be a derivation and let $D_{\mathcal{A}}^1, D_{\mathfrak{A}}^1, D_{\mathcal{A}}^2$ and $D_{\mathfrak{A}}^2$ associated to D as in Corollary 4.9. Then it is not hard to check that if $D_{\mathcal{A}}^1 = \text{ad}_{a^{(2n+1)}}$, then $D_{\mathfrak{A}}^1 = \text{ad}_{a^{(2n+1)}}$ and

$$D_{\mathcal{A}}^2(a) = a \blacktriangle a^{(2n+1)} - a^{(2n+1)} \bullet a,$$

for all $a \in \mathcal{A}$. It follows that $D = \text{ad}_{(a^{(2n+1)}, \beta^{(2n+1)})}$ if and only if $D_{\mathcal{A}}^1 = \text{ad}_{a^{(2n+1)}}$ and $D_{\mathfrak{A}}^2 = \text{ad}_{\beta^{(2n+1)}}$ for some $\beta^{(2n+1)} \in \mathfrak{A}^{(2n+1)}$ and $a^{(2n+1)} \in \mathcal{A}^{(2n+1)}$. Thus $\mathcal{A} \rtimes \mathfrak{A}$ is $(2n+1)$ -weakly amenable if and only if \mathcal{A} and \mathfrak{A} are $(2n+1)$ -weakly amenable. A similar argument is true for even n . \blacksquare

Theorem 5.5 *If \mathcal{A} and \mathfrak{A} are cyclicly amenable and $\langle \mathcal{A}^2 \rangle = \mathcal{A}$, then $\mathcal{A} \rtimes \mathfrak{A}$ is cyclicly amenable.*

Proof. Suppose that $D : \mathcal{A} \rtimes \mathfrak{A} \rightarrow (\mathcal{A} \rtimes \mathfrak{A})^{(1)}$ is a cyclic derivation. Then, there exists $D_{\mathfrak{A}}^1 \in Z^1(\mathfrak{A}, \mathcal{A}^{(1)})$, a bounded linear map $D_{\mathcal{A}}^2 : \mathcal{A} \rightarrow \mathfrak{A}^{(1)}$ and two cyclic derivations $D_{\mathcal{A}}^1 \in Z^1(\mathcal{A}, \mathcal{A}^{(1)})$, $D_{\mathfrak{A}}^2 \in Z^1(\mathfrak{A}, \mathfrak{A}^*)$ such that $(D_{\mathcal{A}}^2)^*(\beta) + D_{\mathfrak{A}}^1(\beta) = 0$ and

$$D((a, \beta)) = (D_{\mathcal{A}}^1(a) + D_{\mathfrak{A}}^1(\beta), D_{\mathcal{A}}^2(a) + D_{\mathfrak{A}}^2(\beta)),$$

for all $a \in \mathcal{A}$ and $\beta \in \mathfrak{A}$. Moreover, by assumption there exists $a^{(1)} \in \mathcal{A}^{(1)}$ and $\beta^{(1)} \in \mathfrak{A}^{(1)}$ such that $D_{\mathcal{A}}^1 = \text{ad}_{a^{(1)}}$ and $D_{\mathfrak{A}}^2 = \text{ad}_{\beta^{(1)}}$. Hence, for each $a, b \in \mathcal{A}$, we have

$$\begin{aligned} \langle D_{\mathfrak{A}}^1(\beta), ab \rangle &= -\langle (D_{\mathcal{A}}^2)^*(\beta), ab \rangle \\ &= -\langle D_{\mathcal{A}}^2(ab), \beta \rangle \\ &= -\langle D_{\mathcal{A}}^1(a) \bullet b + a \blacktriangle D_{\mathcal{A}}^1(b), \beta \rangle \\ &= -\langle \text{ad}_{a^{(1)}}(a) \bullet b + a \blacktriangle \text{ad}_{a^{(1)}}(b), \beta \rangle \\ &= -\langle a \triangle a^{(1)} - a^{(1)} \circ a, b \cdot \beta \rangle - \langle b \triangle a^{(1)} - a^{(1)} \circ b, \beta \cdot a \rangle \\ &= -\langle a \triangle a^{(1)}, b \cdot \beta \rangle + \langle a^{(1)} \circ a, b \cdot \beta \rangle - \langle b \triangle a^{(1)}, \beta \cdot a \rangle + \langle a^{(1)} \circ b, \beta \cdot a \rangle \\ &= \langle a^{(1)} \circ a, b \cdot \beta \rangle - \langle b \triangle a^{(1)}, \beta \cdot a \rangle \\ &= \langle a^{(1)}, (ab) \cdot \beta \rangle - \langle a^{(1)}, \beta \cdot (ab) \rangle \\ &= \langle \beta \blacktriangle a^{(1)}, ab \rangle - \langle a^{(1)} \bullet \beta, ab \rangle \\ &= \langle \beta \blacktriangle a^{(1)} - a^{(1)} \bullet \beta, ab \rangle. \end{aligned}$$

This together with the fact that $\langle \mathcal{A}^2 \rangle = \mathcal{A}$, implies that

$$D_{\mathfrak{A}}^1(\beta) = \beta \blacktriangle a^{(1)} - a^{(1)} \bullet \beta,$$

for all $\beta \in \mathfrak{A}$. The same argument shows that

$$D_{\mathcal{A}}^2(a) = a \blacktriangle a^{(1)} - a^{(1)} \bullet a,$$

for all $a \in \mathcal{A}$. We now invoke Proposition 4.5 to conclude that $D = \text{ad}_{(a^{(1)}, \beta^{(1)})}$. \blacksquare

Recall from [13, Example 2.5] that if $0 \neq \mathcal{C}$ is a Banach algebra with zero algebra product, then \mathcal{C} is cyclicly amenable if and only if $\dim \mathcal{C} = 1$. In particular, cyclic amenability of the Banach algebra \mathcal{C} does not implies that $\langle \mathcal{C}^2 \rangle = \mathcal{C}$.

Now, we are in position to show that the condition $\langle \mathcal{C}^2 \rangle = \mathcal{C}$ in Theorem 5.5 can not be omitted.

Example 5.6 Let $\mathcal{A} = \mathbb{C} = \mathfrak{A}$ with zero algebra products. Then by [13, Example 2.5] \mathcal{A} and \mathfrak{A} are cyclicly amenable. In particular, $\langle \mathcal{A}^2 \rangle \subsetneq \mathcal{A}$. But, again by another application of [13, Example 2.5], $\mathcal{A} \rtimes \mathfrak{A}$ is not cyclicly amenable. This is because of, $\mathcal{A} \rtimes \mathfrak{A}$ is a commutative Banach algebra with zero algebra product and dimension 2.

We note that with an argument similar to the proof of Theorem 5.3 one can prove the following result. The details are omitted.

Theorem 5.7 *If $\mathcal{A} \rtimes \mathfrak{A}$ is cyclicly amenable, then the following assertions hold.*

- (i) \mathfrak{A} is cyclicly amenable.
- (ii) If $(\mathcal{A}, \mathfrak{A})$ has the property (\mathbb{H}_1) , then \mathcal{A} is cyclicly amenable.

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